Trig 4.5a: Solving Trig Equations, Part 1

It is not unusual in different areas of the sciences to run into an equation that involves a trig function, and for us to be expected to solve that equation. We will get to that soon enough, but first let's talk about solving equations in general.

When we run into an equation, a first thing that is helpful for us to do is to identify the type of equation. In particular, we want to identify whether the equation is linear or not. As we proceed, keep in mind that we are working with single variable equations only. That is, we won't have both x & y in the equation.

<u>Linear Equations</u>
Generally speaking, a linear equation is one that is equivalent to (you might say "can be rearranged to") the form $ax+b=0$, where a is not allowed to be zero. The key here is that every time x appears, it must be to the first power. That rules out x in the denominator, and x multiplied by another x anywhere. The "first power" part also rules out "inside a root."

Example 1: Which of the following are linear? (Answers on the next page.)

a) $3x + 2 = 4x - 1$ c) $x^2 = x + 1$

b) $x(x+1) = 2x-3$ d) $3(x+1) = x(3-7)$

This type of linear is technically called "linear in x." That means, if any other variable appeared, it wouldn't matter how it appeared (squared, denominator, etc), as long as the x parts followed the rules.

We can also do linear in other things, such as x^2 or $\sin(x)$. That would mean, in the x^2 case, every place the x^2 appears, it must appear to no other power than 1, it must not be in a denominator, and it must not be multiplied by any other x's.

Example 2: These are all either linear in some way, or not linear at all. Can you identify how?

a) $x^2 + 3 = 15$ c) $x(x+1) = 2x-3$

b) $3\sin(x) = 1$ d) $x^2 = x + 1$

Answers to the previous page

Example 1. (a) is linear;
(b) is not linear (multiply out the parentheses)
(c) is not linear (It has a square)
(d) is linear

Example 2. (a) is linear in x^2;
(b) is linear in sin(x)
(c) is not linear (After you multiply out the parentheses, It has x & x^2)
(d) is not linear (It has x & x^2)

Solving linear equations

Solving an equation means finding a value of the variable that makes the equation true. That is, finding what x must be. Once you've identified an equation as linear, there is a straight-forward approach to solving that equation.

> Move all of the variable pieces to one side,
> and all of the non-variable pieces to the other side.

You may have to manipulate some things in the equation first, such as multiplying out any parentheses, before you are able to do that.

Example 3: Solve $4x+7=6x-2$

We recognize that this equation is linear (in x), so we move the x pieces to one side and the non-x pieces to the other side.

$$4x + 7 = 6x - 2$$
$$-6x - 7 \quad -6x - 7$$

$$\Rightarrow \quad 4x - 6x = -2 - 7$$

Now we simplify

$$\Rightarrow \quad -2x = -9$$

Then we isolate x, by dividing both sides by -2

$$\Rightarrow \quad x = 9/2$$

That's our solution!

Example 4: For you to try: Solve $5(x-2) = 3x + 8$

Answer to the previous page

Example 4: Solution is $x = 9$

If we have an equation that is linear in something else (instead of just x), then we can follow this same procedure, though there will probably be an extra step or two at the end where we isolate the variable.

Example 5: Solve $x^2 + 3 = 15$
We observe that this equation is linear in x^2, so we isolate x^2 just like we isolated x in the previous example.

$$\begin{aligned} x^2 + 3 &= 15 \\ -3 \quad &\quad -3 \end{aligned}$$

$$\Rightarrow \quad x^2 = 12$$

Now the final step is to take the square root of both sides. Remember, when WE take a square root, we must allow for the possibility it could be ±.
So

$$x = \pm\sqrt{12}$$

<u>So now we move onto linear equations involving trig functions.</u>

Example 6. Solve $\sin(x) + 1 = 0$
We observe that the equation is linear in sin(x), so we move the sines to one side and everything else to the other, to get:

$$\sin(x) = -1$$

At this point we have solved the linear equation, but now we must (as we mentioned earlier) actually find what x equals. This is where we come back to our trig studies. We need to find every value x where $\sin(x) = -1$. There are several ways we could think through this, but I think the easiest is with the unit circle. Remembering that the sine is the y-value as we go around the unit circle, let's go once around that circle and find every time the y-value hits -1. How many times does it do it? Only once, at the very bottom of the circle. The angle for that spot is $3\pi/2$, so we know that one answer is $x = 3\pi/2$

But there are more answers, right? Because we know that sine is a 2π-periodic function, we know that we can add (or subtract) 2π on as many times as we want to get another solution. Here's a shortcut way to write that:

$$x = 3\pi/2 + 2\pi n$$

When we write this notation, it is assumed that n is a whole number, including positives, negatives and zero.

Some comments before we do more examples:
a) We will do that $2\pi n$ thing every time, except when our trig function is tangent or cotangent, in which case we just do πn. (Do you understand why?)
b) If we are using sine or cosine, as we move around the unit circle once, how many times do we hit the value 1? What about -1? What about values above 1? Below -1? Between 1 & -1?

Here's the summary:

We hit every value between 1 & -1 twice.
We hit the values 1 & -1 only once per cycle.
We never hit the values above 1 or below -1.

So, for example, if our equation was $\sin(x)=2$, there will be no solutions, since sine cannot equal 2. And if the equation was $\cos(x)=0.45$, there will be two solutions (per cycle), and, in fact, those solutions will be in the 1st & 4th quadrant, since that is where cosine is positive.

Comments continued:
c) If we are using secant or cosecant, we hit every value above 1 or below -1 twice, we hit the values 1 & -1 once, and we hit the values between 1 & -1 never, per cycle. Think of how the graphs of these two functions look to help you remember this.

d) If we are using tangent or cotangent, we hit every value EVERY VALUE! exactly once per cycle. Think of how the graphs of these two functions look to help you remember this. Tangent starts at $-\infty$ and moves up to $+\infty$, hitting every height along the way one time.

Example 7 Solve $2\cos(x) - 1 = 0$.
Linear in cosine, so $\Rightarrow$ $2\cos(x) = 1$ $\Rightarrow$ $\cos(x) = 1/2$.
How many times per cycle does cosine hit 1/2? Twice, in quads I & IV.

Let's find the quad I answer first. Think of the 30-60-90 triangle. Cosine would equal 1/2 for the 60° angle, which is $\pi/3$.

<u>We are always doing these in radians.</u>

Now we need to find a 4th quadrant angle where cosine equals 1/2. Well, our two answers must have the same reference angle, and the reference angle is always the first quadrant angle, so $\pi/3$. We need to find a 4th quadrant angle with reference angle $\pi/3$. There are many, and it doesn't matter which one you choose, so let's choose $-\pi/3$. There's our two angles for one cycle.
Finally, we tack on the $2\pi n$ thing. So our final answer is:

$$x = \pi/3 + 2\pi n \quad \text{or} \quad x = -\pi/3 + 2\pi n.$$

Example 8: Solve $\tan(x) - 1 = 0$
Linear in tan(x), so: $\tan(x) = 1$.

How many times does tangent hit 1 per cycle?	Once
What is one quadrant where tangent is positive?	First
What angle in the first quadrant has tangent equal to 1? (Think about the 45-45-90 triangle.)	45°, which is $\pi/4$
How periodic is tangent?	π

So our answer is: $x = \pi/4 + \pi n$

Example 9: Solve $\sec^2 x - 2 = 0$
Linear in $\sec^2 x$, so: $\sec^2 x = 2$
$\Rightarrow$ $\sec(x) = \sqrt{2}$ and $\sec(x) = -\sqrt{2}$
We've broken the problem into two linear problems. Let's solve those.
$\sec(x) = \sqrt{2}$, which means $\cos(x) = 1/\sqrt{2}$. Cosine hits this value twice per cycle, in quadrants I & IV, and $1/\sqrt{2}$ is from the 45-45-90 triangle, so $x = \pi/4$ in Quad I and $x = 7\pi/4$ in Quad IV.

$\sec(x) = -\sqrt{2}$, which means $\cos(x) = -1/\sqrt{2}$. Cosine hits this value twice per cycle, in quadrants II & III, and $1/\sqrt{2}$ is from the 45-45-90 triangle, so $x = 3\pi/4$ in Quad II and $x = 5\pi/4$ in Quad III.

Putting it all together we get:

$x = \pi/4 + 2\pi n$
$x = 3\pi/4 + 2\pi n$
$x = 5\pi/4 + 2\pi n$
$x = 7\pi/4 + 2\pi n$

<u>Your homework for this lesson:</u>

Section 4.5: You should be able to do all of #1-10, but do at least 2,4,6,10, and email your answers to me.