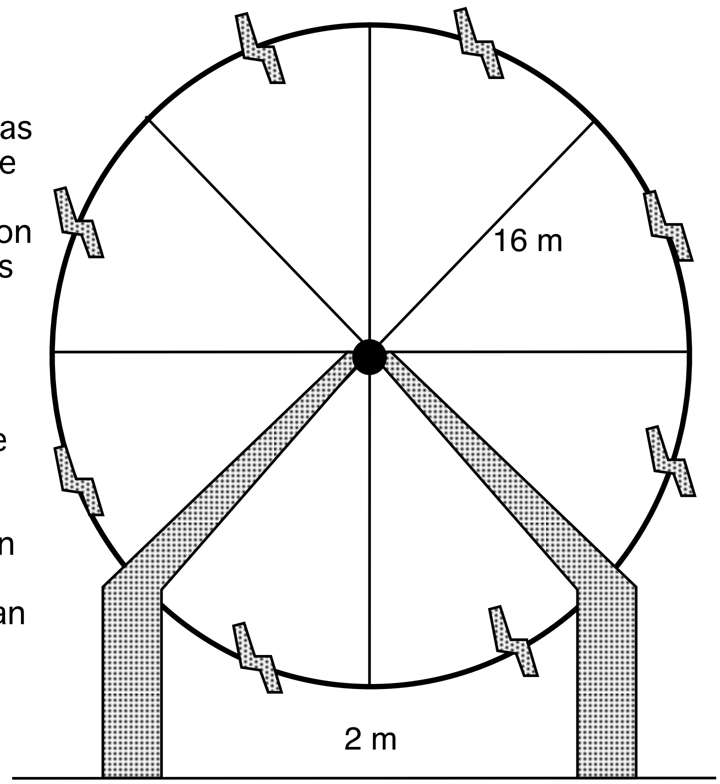
Examples of Trig Functions

Today we look at a few examples of items that can be represented by trigonometric functions. You need not memorize any of these equations or functions, but understand what they are saying and be able to work with them.

1) Motion of a Ferris Wheel



The giant ferris wheel pictured to the right has a radius of 16 meters, and the bottom of the wheel passes 2 meters above the ground. If the ferris wheel makes one complete revolution every 20 seconds, find an equation that gives the height above the ground of a person on the ferris wheel as a function of time.

Basically we need to find the constants A, B, C & D so the following function describes the motion:

$$f(t) = A \sin(Bt + C) + D.$$

To make things a little simpler, let's ignore an idea of "phase shift." We will just start the wheel in such a way that phase shift is not an issue.

Q1. What does the "no phase shift" decision tell us about A, B, C and/or D?

Q2. What is the vertical shift of this sine wave, and what does that tell us about A, B, C, and/or D?

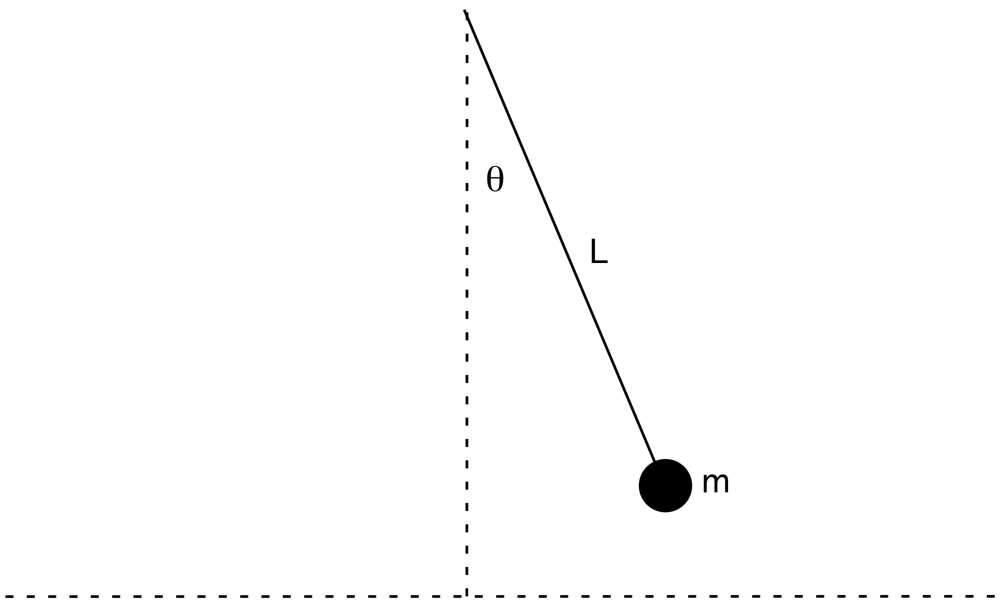
Q3. What is the amplitude of this sine wave? A, B, C, and/or D?

Q4. What is the period of this sine wave? A, B, C, and/or D?

Q5. Put it all together: What is our function?

Q6. Another ferris wheel has a radius of 12 feet, completes a revolution in 10 seconds, and actually dips into the ground 4 feet when it comes down. Construct the function for this one.

2) Oscillation of a Pendulum



A pendulum has a bob on the end. It is set into motion by being pulled to the right (or being pushed to the left) a certain angle and then being released. The resulting angle is sinusoidal (i.e. has a graph like a sine or cosine wave).

Symbols: L = length of the pendulum (usually in meters or feet)
m = mass of the bob (usually in kg or lb)
θ = angular displacement of the pendulum (in degrees)
Note: displaced to the right means $\theta>0$
displaced to the left means $\theta<0$
t = amount of time since the bob was released (usually in seconds)

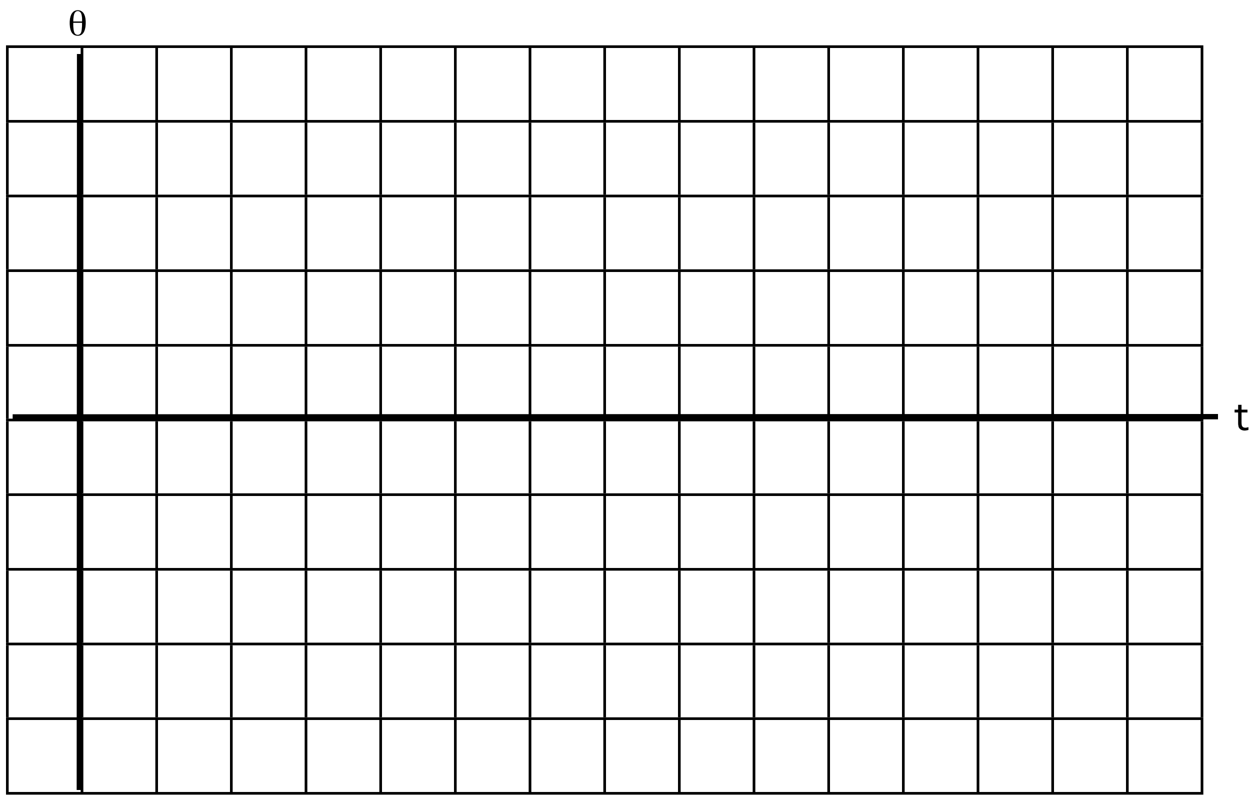
Q7. One or more of those symbols are variable. Which would that be?

Q8. If there is more than one variable, do those variables rely on one another?
(i.e. Is one a function of the other?)

If you answered Q8 correctly, you should have had that θ was a function of t. That is, as time advances, the angle θ is based upon the time that has elapsed.
In other words, θ could be more accurately denoted by: $\theta(t)$.

Our job: To construct the function $\theta(t)$.

Q9. An intuition check. Suppose we have the pendulum described earlier. Suppose we pull the bob of that pendulum to an initial angle of 10°, then release the bob (at an initial time of 0 seconds). Suppose it takes 4 seconds for the bob to get back to its starting position. On the x-y graph below, sketch a rough picture of the function $\theta(t)$.



Q10. More intuition check. Suppose we push the bob to an initial angle of -15°. Again, it takes 4 seconds to get back to the starting point. Sketch a rough picture of the function $\theta(t)$ on the same x-y graph above.

Possibly it is easy for you to see that cosine waves make sense for the above graphs. In particular, you might realize:

- under ideal conditions, the graph is periodic. That is, once it completes a cycle, it's going to complete another cycle identical to the first.
- the angle changes slowly at first, then changes more rapidly when the bob is at the bottom of the pendulum, and then slows down again as it approaches the extreme.
- the angle never exceeds (in a positive or negative sense) the initial angle.

Q11. Perhaps those reasons above can convince you that this is either a sine wave or a cosine wave. What additional feature tells you that this would have to be a cosine wave, not a sine wave?

Okay, so now we know that the function must be of the form $\theta(t) = A\cos(Bt)$.

Q12. Back to Q9 & Q10. Using the given data for those problems, fill in the chart below and then identify the particular $\theta(t)$ for each case.

	Q9	Q10
initial angle	10°	-15°
time per cycle	4 sec	4 sec
Amplitude		
Period		
Constant A		
Constant B		

For Q9: $\theta(t) =$

For Q10: $\theta(t) =$

Here's what we have for pendulums:

$\theta(t) = A \cos(Bt)$

where

A equals initial angle
B equals 2π/period

Beyond our scope, it is also true that:

B equals sqrt(g/L)
(gravitational constant of accel/length of pendulum)

Note: In the metric system, g = 9.8 m/s^2.
In the American system, g = 32 ft/s^2.

Important note: The period of the pendulum depends on the length of the pendulum, but not on the mass of the bob.

Q13. Construct the pendulum function for a pendulum of length 2 meters and initial angle of 5°.

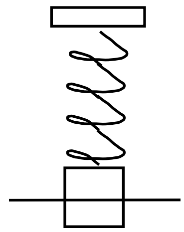
Q14. What is the period for the pendulum in Q13?

Q15. Knowing that B = 2π/period, and also that B = sqrt(g/L), find an expression for the period as a function of L.

Q16. What is the period for a pendulum of length 10 feet?

Q17. If we want to build a pendulum with period of 2 seconds, how long must the pendulum be?

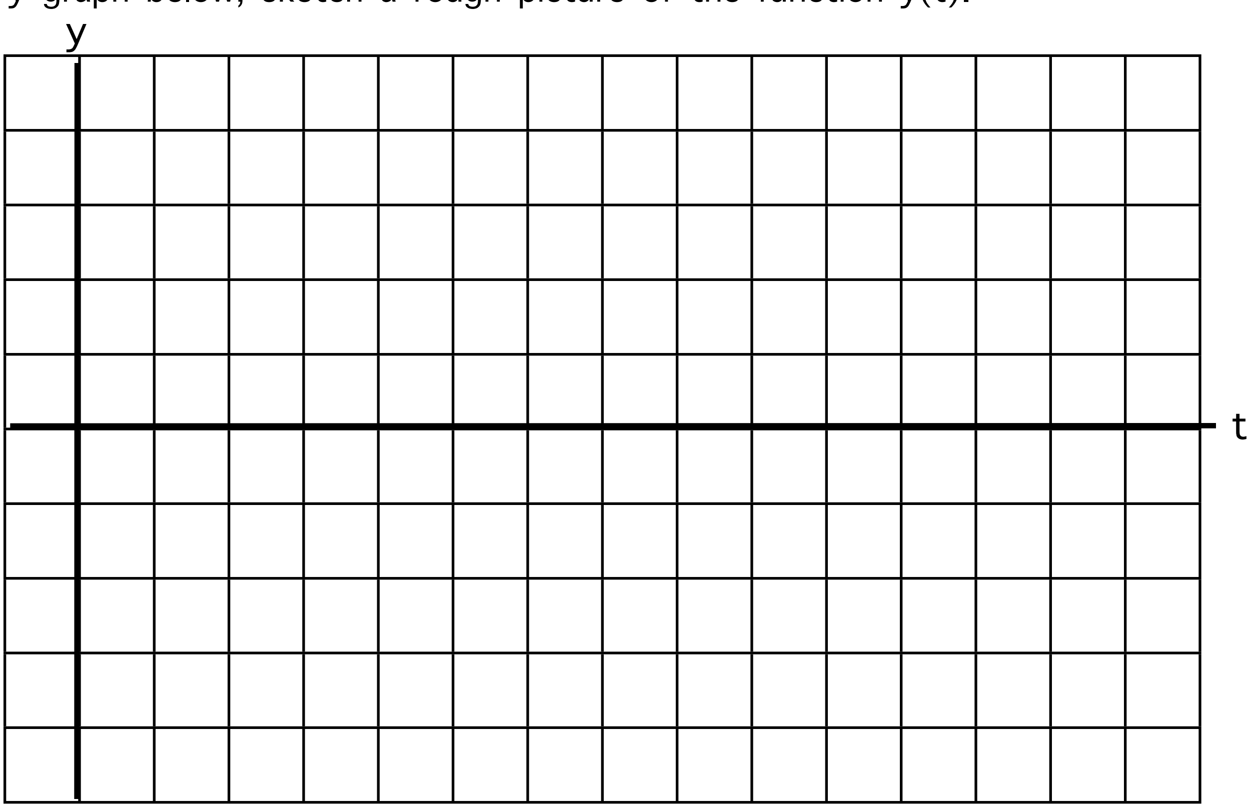
3) Vibrating spring



Now we have a spring attached to the ceiling, and a block (with mass m) attached to the bottom end. The spring is (for the moment) sitting at equilibrium. Let's think of this equilibrium location as y = 0. We're going to pull the block downward (we will think of that as a positive y-value), and then release it.

Our job: To construct the function y(t).

Q18. An intuition check. Suppose we stretch the spring out to 4 inches and release it. Suppose it takes 2 seconds for the spring to return to the initial (4 inches) point. On the x-y graph below, sketch a rough picture of the function y(t).



Q19. If we compress the spring by 2 inches, sketch the graph. (Assume again that it takes 2 seconds to get back to the original compressed length.)

Here's what we have this time:

$y(t) = A \cos(\omega t)$

where

A equals initial stretched amount (so negative if compressed)
ω equals 2π/period (this is called the natural angular frequency)
(This is the greek letter "omega")

Beyond our scope, it is also true that:

ω equals sqrt(k/m)

where

k = the spring constant of the spring
m = mass of the block at the bottom

Note: It is sometimes useful to know that weight = mass * g.

Q20. A spring is stretched out to 3 inches, and released. It returns in 2 seconds. Find the function.

Note: Once the ω is determined for a spring/block combination, it stays the same, regardless of how much the spring is stretched or squished (within reason). The same is true of k.

What does "within reason" mean? Clearly if you stretch it too far, you go beyond the "elastic limit," and the spring "deforms." Same for "squishing" too far.

Q21. The same spring as in Q20 is “squished” by 4 inches, then released. Find the function.

Q22. If the weight of the block was 5 pounds, then what is the spring constant for this spring?

Some extra facts about springs:

- If we know that a spring follows the function $y = A\cos(\omega t)$, then we can use calculus to measure the velocity of the spring to be $v(t) = A\omega \sin(\omega t)$.

- When the conditions are right to cause a spring to slow down, we say that the spring experiences damping. This would happen, for example, if the spring was operating under water, since the water would cause greater resistance to the block than air would. The amount of damping a spring experiences is sometimes called the attenuation.

- Sometimes a spring experiences forced motion. This occurs when some kind of extra force (separate from the spring & gravity forces) is added into the block. For example, if every time the block reached its extreme you chose to whack it back with a baseball bat, that would be forced motion. When the forcing function is periodic, it is sometimes called an exciter function. Such an exciter function would have its own angular velocity.

- When a spring experiences forced motion, and the natural angular velocity of the spring exactly matches the angular velocity of the exciter function, then the spring experiences what is known as resonance. The two frequencies work together to cause the amplitude of the spring to grow and grow, until finally the spring exceeds its elastic limit. There are many real world examples of the effects of resonance. Three such examples are the Tacoma Narrows Bridge (Puget Sound, Washington) in late 1940, the Hyatt Regency Hotel (Kansas City) in the early 1980s, and the Broughton Bridge (Manchester, England) in 1831.

If you want to explore more about springs, there is an excellent demonstration at the website http://www.walter-fendt.de/ph11e/index.html. (This website is also linked from our class website.) Scroll down the page to the “Oscillations and Waves” section, then find the link for “Forced Oscillations (Resonance).” You can set the level of attenuation and the angular frequency of the exciter function, among other things. Make sure you choose “Elongation diagram” from the options to the right.