Differential Equations - Lesson 8.2A

We saw last time that the general linear system of differential equations is:

$x' = a_1x + b_1y + f_1(t)$
$y' = a_2x + b_2y + f_2(t)$

a_1, a_2, b_1, b_2 are all constants.

Then we saw that the entire system could be written in matrix form:

Using $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$

we have $X' = AX + F$.
We call this NHS for Linear Nonhomogeneous System

We also have the corresponding LHS: $X'=AX$.

We agreed that each solution to this system is a pair of functions, which we will write as a 2x1 matrix, like X above.

And we agreed that for an nxn system we need to find n linearly independent solutions. For us that will usually be a 2x2 system and 2 lin indep solutions.

We attempt to solve LHS: $X' = AX$

We assume there exists a soln of the form $X = \begin{bmatrix} c_1e^{\lambda t} \\ c_2e^{\lambda t} \end{bmatrix}$.

Those c's are constants (to be determined), and the λ is another constant (to be determined). It makes sense that both parts must have the same λ, since both DEs require both functions added together to get the derivative of one or the other of them. If they had different λ's, they would be independent and not combine correctly.

To simplify that assumed solution, we're going to factor out the $e^{\lambda t}$, which leaves a constant matrix C with just the c_1 & c_2. We will call this $C\, e^{\lambda t}$. That is our assumed solution: $X = Ce^{\lambda t}$

We plug our assumed solution $X = Ce^{\lambda t}$ into the LHS $X' = AX$.

Here we go: $C\lambda e^{\lambda t} = ACe^{\lambda t}$

$\Rightarrow$ $C\lambda e^{\lambda t} - ACe^{\lambda t} = 0$ (moved all to the left)

$\Rightarrow$ $ACe^{\lambda t} - \lambda Ce^{\lambda t} = 0$ (just turned it around, and moved the constant λ through the matrix C)

$\Rightarrow$ $(A-\lambda I)Ce^{\lambda t} = 0$ (factored out $Ce^{\lambda t}$ - detail below*)

$\Rightarrow$ $(A-\lambda I)C = 0$ (divide both sides by $e^{\lambda t}$)

*The detail. When you factor the $Ce^{\lambda t}$ out, you leave $A-\lambda$ behind. But A is a matrix and λ is a scalar, and they cannot be subtracted. Instead we insert what is called the "identity matrix" I, which is (in the 2x2 case)

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

You might just trust me that this is what we need to do, but I can supply some other examples or references if you want to be more convinced.

Our job is to find C and/or λ to make $(A-\lambda I)C = 0$

We can clearly see that if we take $c_1=0$ and $c_2=0$, then that will solve the equation. But we've seen before that if 0 & 0 are part of the fundamental set of solutions, the set cannot be linearly independent, and thus not really a fundamental set of solutions. We must find <u>another</u> solution than 0 & 0.

That is, we want to know if $(A-\lambda I)C = 0$ has <u>only one solution</u> (0 & 0) or <u>more than one solution</u>.

<u>A property of Linear Algebra:</u>

PX=0 has only one solution if $\det(P) \neq 0$, (det = determinant)
but PX=0 has more than one solution if $\det(P) = 0$.

If we want to find a nonzero solution, we must find the value of λ that causes

$$\det(A-\lambda I) = 0$$

Then we find a matrix C that goes with λ
and then we have $Ce^{\lambda t}$ as a soln to the DE.

Terminology:

A is called the <u>coefficient matrix</u> of the system

$\det(A-\lambda I)=0$: the <u>characteristic equation</u> of A

Values of λ that cause $\det(A-\lambda I)=0$ are called <u>eigenvalues</u> of A
Pronounced EYE-GUN-Values

Vectors or matrices C_λ that correspond to eigenvalues are called <u>eigenvectors</u> of A

Example: Solve

$$x' = -x + 6y$$
$$y' = x - 2y$$

$$A = \begin{bmatrix} -1 & 6 \\ 1 & -2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad A-\lambda I = \begin{bmatrix} -1-\lambda & 6 \\ 1 & -2-\lambda \end{bmatrix}$$

$\det(A-\lambda I) = (-1-\lambda)(-2-\lambda) - 6 = \lambda^2+3\lambda+2-6 = \lambda^2+3\lambda-4$

We want this equal to zero, so we set $\lambda^2+3\lambda-4 = 0$ <-- characteristic eqn

$(\lambda+4)(\lambda-1) = 0 \quad \Rightarrow \quad \lambda=1$, or $\lambda=-4$ <-- eigenvalues

Now we need to find eigenvectors to go along with our eigenvalues. We need to do it separately for each eigenvalue. It turns out there are infinitely many eigenvectors for each eigenvalue, but they are all multiples of one another. So we will just find one, because eventually we will multiply by an arbitrary constant anyway.

<u>For $\lambda=1$</u>,we must find a non-zero C to make $(A-1I)C = 0$. In matrix form, that says:

$$\begin{bmatrix} -1-1 & 6 \\ 1 & -2-1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we multiply out the left side of that second form, we get

$$-2c_1 + 6c_2 = 0 \text{ and } 1c_1 + -3c_2 = 0.$$

We need to solve these equations simultaneously, but (a feature of being a 2x2 problem) these two equations are equivalent. So we only need to find one solution to one of them. That solution will also work in the other one.

Let's do the second one (no reason). Can you find two numbers c_1 and c_2 that make the second one true? (Answer on the next page.)

Answer from the previous page: One solution is $c_1=3$ and $c_2=1$. You might have found a different one, but it should be a multiple of this one, such as 6 & 2, or -3 & -1, or many others. Let's stick with 3 & 1. (You might think of it as reduced.)

So, if we use the c-values 3 & 1, we get the solution

$$X = Ce^{\lambda t} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{t}$$

So that means $x=3e^t$ and $y=1e^t$ solve the system, but let's stay in matrix form.

Now for $\lambda=-4$.

$$\begin{bmatrix} -1+4 & 6 \\ 1 & -2+4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Top equation is $3c_1+6c_2=0$, which has a solution of 2 and -1. So

$$X = Ce^{\lambda t} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-4t}$$

Now we construct the <u>general solution</u>, the linear combination of the two solutions we found.

$$X = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-4t}$$

Let's add some initial conditions, like $X(0) = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$. Plugging in we get

$$\begin{bmatrix} 7 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$ (the e-parts are both e^o, so 1)

Which means $7 = 3c_1 + 2c_2$ and $-1 = c_1 - c_2$. Solving simultaneously, we get $c_1=1$ and $c_2=2$.

So our <u>solution to the IVP</u> is $X = 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{t} + 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-4t}$.

Now that we know what we are doing, here's one all the way through.

Example: Solve

$$x' = x + 2y$$
$$y' = 4x + 3y$$

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \qquad A\text{-}\lambda I = \begin{bmatrix} 1\text{-}\lambda & 2 \\ 4 & 3\text{-}\lambda \end{bmatrix}$$

$\det(A\text{-}\lambda I) = (1\text{-}\lambda)(3\text{-}\lambda) - 8 = \lambda^2\text{-}4\lambda+3\text{-}8 = \lambda^2\text{-}4\lambda\text{-}5$

Set $\lambda^2\text{-}4\lambda\text{-}5 = 0$ <-- characteristic eqn

$(\lambda\text{-}5)(\lambda+1) = 0 \Rightarrow \lambda=5$, or $\lambda=-1$ <-- eigenvalues

<u>For $\lambda=5$</u> we have:

$$\begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Top equation is $-4c_1 + 2c_2 = 0$, and a solution to that is 1 & 2.

<u>For $\lambda=-1$</u> we have:

$$\begin{bmatrix} 1+1 & 2 \\ 4 & 3+1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Top equation is $2c_1+2c_2=0$, which has a solution of 1 and -1. So

$$X = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$$

Much quicker this time! (I hope)

A shortcut when you are trying to solve the equation leading to the eigenvector:

If you are trying to solve, say, $5c_1+7c_2=0$, one solution is to take the opposite coefficient as your c-value, but you must negate one of them. That is, use the coefficient of c_1 to get a value for c_2, and the coefficient of c_2 to get the value for c_1, but put a minus sign on one of them. So $c_1=7$ and $c_2=-5$.

If you are trying to solve $3c_1-4c_2=0$, keep in mind that the negative sign is part of the coefficient of c_2. So we can take $c_1=4$ and $c_2=3$. (We negate the -4 to get the 4.)

This shortcut will be much more important when we get to complex eigenvalues, but it's pretty handy now, too.

Your homework for this part:
- I recommend you try #1,3,5,13, and check your answers in the back.
- Then do #2 & #4, and submit your answers to me via email. No need to show any work. Just the answers. And don't worry about notation. You can just say something like "lambda = 5 and the c-values are 3 & 2".