# Differential Equations - Lesson 8.2B

Last time we solved this 2x2 system $x' = -x + 6y$
$y' = x - 2y$

We saw that it had an eigenvalue of $\lambda=1$ with eigenvector of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$,

and an eigenvalue of $\lambda=-4$ with eigenvector of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. This leads to the general

solution $X = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 2 \\ -1 \end{bmatrix}e^{-4t}$ . We're going to identify some patterns to these solutions.

We know that we can determine $c_1$ & $c_2$ from initial conditions. Let's suppose we have initial conditions that translate into $c_1=1$ and $c_2=0$. Then our solution would be $X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{t}$ , which means $x(t) = 3e^{t}$ and $y(t) = e^{t}$. Is it clear that, in this case, the x-value is always 3 times the y-value? So, if we were to sketch on an x-y graph all of the points that meet this condition, we would sketch a half-line with slope 1/3 that goes through the origin, so half of the line $y=(1/3)x$. (Only a half-line because it is clear that x & y are both positive, so we are in the 1st quadrant only.) We would call this a <u>straight-line solution</u>. If you start on this line, you stay on the line. If you start off of the line, you stay off of the line. In fact, you cannot even cross the line. (You know why, don't you? Uniqueness of solutions!)

<u>Here's some questions for you to answer</u>. I'll show my answers on the next page.

1) If we are on that half-line above, as t increases, which direction will we be moving along that half-line: toward the origin or away from the origin?

2) Suppose our initial conditions translate into $c_1=-1$ and $c_2=0$. What straight-line solution does this lead to? And do we move toward the origin or away from the origin?

3) Suppose our initial conditions translate into $c_1=0$ and $c_2=1$. What straight-line solution does this lead to? Toward the origin or away from the origin?

4) Suppose our initial conditions translate into $c_1=0$ and $c_2=-1$. What straight-line solution does this lead to? Toward the origin or away from the origin?

Answers to the previous page:

1) Moves away from the origin, because x= $3e^t$ gets bigger and bigger as t increases.
2) The same line as the previous case: y = (-1/-3)x = (1/3)x, but in the 3rd quadrant this time. It moves away from the origin, because x= $-3e^t$ gets more negative as t increases.

3) Our solution this time is $X = \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-4t}$ , which means $x(t) = 2e^{-4t}$ and

$y(t) = -e^{-4t}$. The x-value is always -2 times the y-value, so we are on the half-line y = (-1/2)x. We are in the 4th quadrant, because x>0 and y<0. As t increases, the e-part goes to 0, so we are moving toward the origin.
4) The same line as the previous case: y = (-1/2)x, but in the 2nd quadrant this time. It moves toward the origin.

-----------------------------------

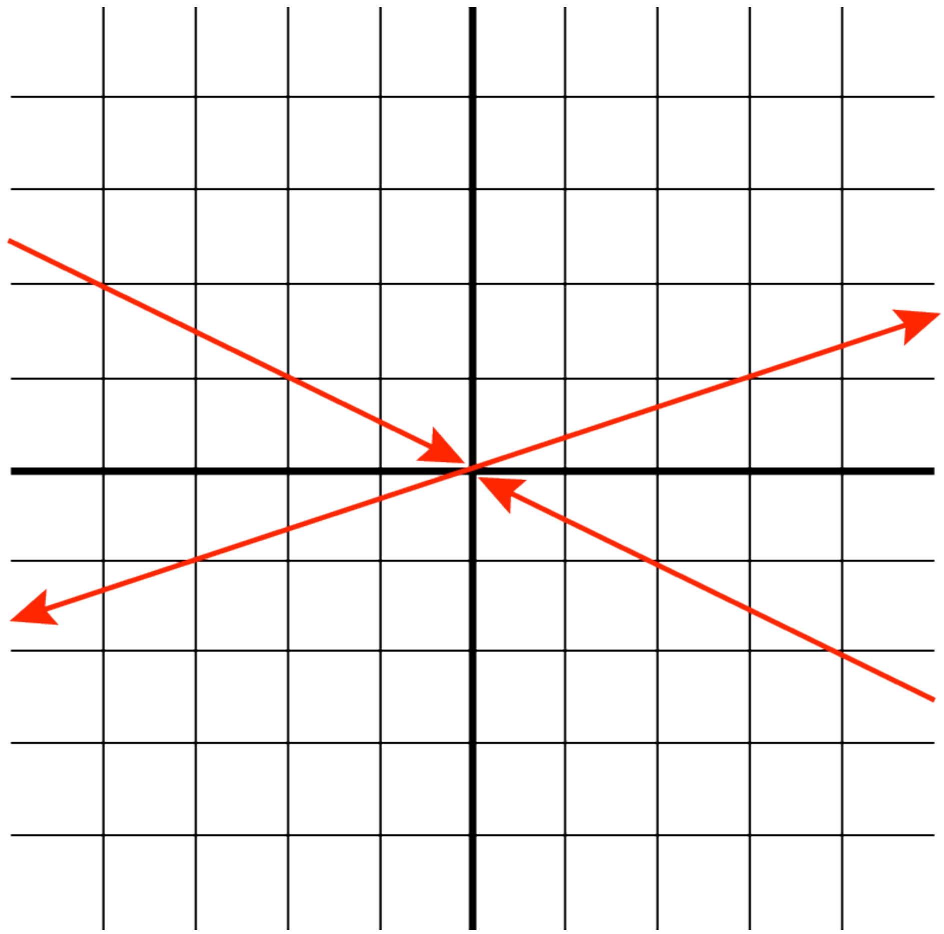
If we piece all that together, we can see that $\lambda=1$ gives a straight-line solution of y = (1/3)x, regardless of the quadrant, and we stay on that line but move away from the origin if we start on the line. And $\lambda=-4$ gives us a straight-line solution of y = (-1/2)x, regardless of the quadrant, and we stay on that line but move toward the origin if we start on the line.

Okay, so here's what we know, and maybe a little that we haven't discovered yet.

- You get a straight-line solution for every distinct real eigenvalue. These systems can have either 0, 1 or 2 distinct real eigenvalues, so they can have 0, 1 or 2 straight-line solutions.

- If you have a distinct real eigenvalue with eigenvector $\begin{bmatrix} a \\ b \end{bmatrix}$, then the straight-line solution would be y = (b/a)x.
- You move away from the origin if the $\lambda$-value is positive, but toward the origin if the $\lambda$-value is negative.
- It may look like the origin itself is part of both answers, but we know that can't be the case! (Uniqueness of solutions!) The only way we can get to the origin is if we start there, and that would require an initial condition of x(0)=0 and y(0)=0, which would translate to $c_1=0$ and $c_2=0$. That means x(t)=0 and y(t)=0. That is, if we start at the origin we stay at the origin. Otherwise we can't hit the origin.

Let's draw our phase plane. So far we have two straight-line solutions

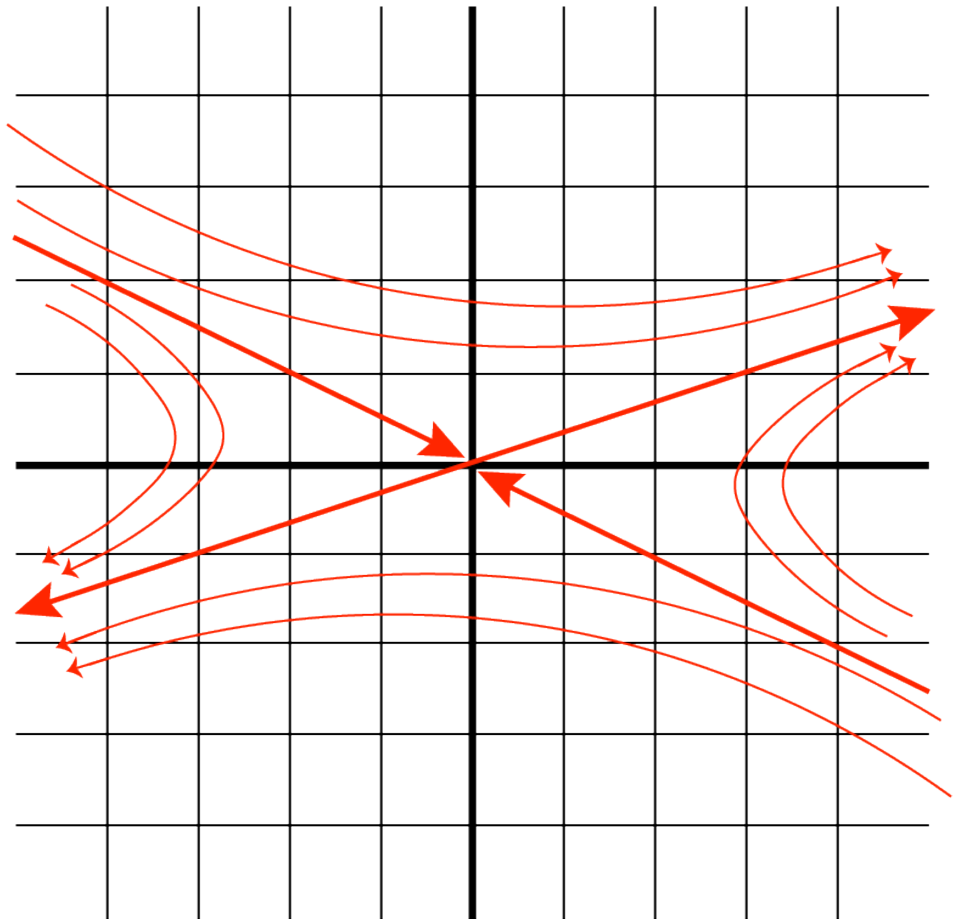


Here's our solution again: $X = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 2 \\ -1 \end{bmatrix}e^{-4t}$

We need to discover what happens elsewhere in the plane. You might notice from the solution that as t gets bigger and bigger, the $e^{-4t}$ gets more and more irrelevant. That is, as t gets bigger, the solution curve becomes more and more like the $e^{t}$ part, so it gets closer to the line for the $e^{t}$ part. On the other hand, as t gets smaller and smaller (toward $-\infty$), the $e^{t}$ part gets irrelevant, and the curve becomes more like the $e^{-4t}$ part.

In each of the four regions, the solution curves are somewhat following the $\lambda=-4$ line, heading generally toward the origin, but then start coming back out, heading toward the $\lambda=1$ line. I don't know if that's clear enough. Can you picture what the solution curves will look like in each region. Maybe try to draw it on scratch paper. I'll draw it on the next page.

There you go. This is what it would look like in the case where there is one positive and one negative eigenvalue.



Now I want you to think about some other cases.

1) Suppose we have a system that yields an eigenvalue of $\lambda=1$ with eigenvector of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and also an eigenvalue of $\lambda=+4$ with eigenvector of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. (Both positive eigenvalues.)

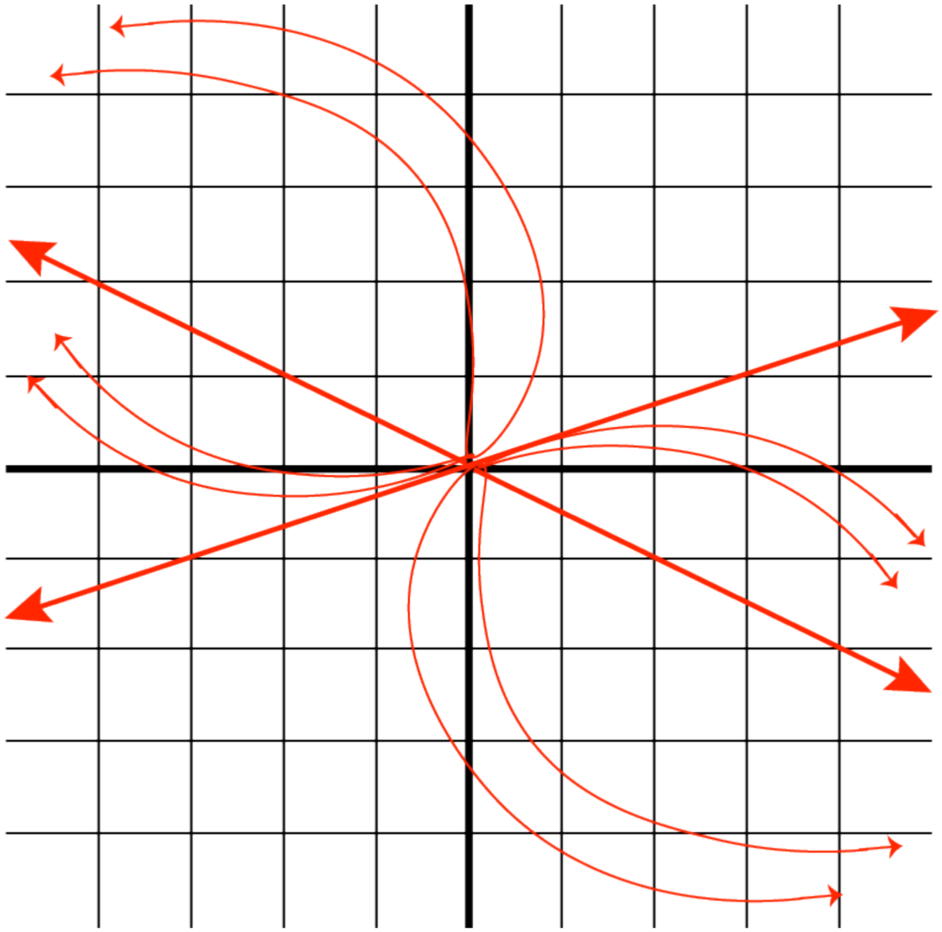
2) Suppose we have a system that yields an eigenvalue of $\lambda=-1$ with eigenvector of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and also an eigenvalue of $\lambda=-4$ with eigenvector of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. (Both negative eigenvalues.)

What would the phase plane look like in these two cases?

Answers from previous page:

1) The straight-line solutions would be the same, but they would both point out from the origin. The lesser eigenvalue (1 this time) would dominate for negative t-values, while the greater eigenvalue (4 this time) would dominate for positive t-values. So the solution curves would come out of the origin, initially following the $\lambda=1$ line, then moving toward the $\lambda=4$ line.

Like this:



2) The straight-line solutions would be the same, but they would both point in toward the origin. The lesser eigenvalue (-4 this time) would dominate for negative t-values, while the greater eigenvalue (-1 this time) would dominate for positive t-values. So the solution curves would come in toward the origin, initially following the $\lambda=-4$ line, then moving toward the $\lambda=-1$ line.

This phase plane would look just like the one just above, except all of the arrows would point the opposite direction.

Homework for today

Section 8.2: Draw (by hand) Phase Portraits for problems 1,2,3 & 4.
Make sure you get those straight-line solutions in the right place, with the arrows pointing the right way. Then draw a couple sample curves in each region, including the arrows.

If you are able, I would like you to scan/photograph your graphs and send them to me. If you are unable to send them to me, you can just describe in an email what you observed.

Try to get this work done by Friday, March 27.