Differential Equations - Lesson 8.2C

We have two more things to consider in our study of LHS problems: the case of real repeated eigenvalues and the case of complex eigenvalues.

Real Repeated Eigenvalues

Example 1. Consider the system $x' = -4x - y$
$y' = x - 2y$

We can set it into matrix form $X' = AX$, and solve to find that we get $\lambda = -3$ & -3. For one of those -3's we get an eigenvector of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which give us a solution of $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$. I hope it is obvious to you that if we took the second -3 and looked for an eigenvector, we would end up with something parallel to the first one. That is, our two solutions would not be linearly independent. So we need to look for another solution.

General Info, not specific to our problem:
Without derivation we claim that, in some cases, a second solution exists of the form $X = Kte^{\lambda t} + Pe^{\lambda t}$
where K & P are appropriately sized matrices. Our job is to find what K & P must be to have such a solution.

How do we do that? We plug $X = Kte^{\lambda t} + Pe^{\lambda t}$ and see what that tells us. We will need to find X'. Using the product rule (and other rules) on the above expression for X we get $X' = Kt\lambda e^{\lambda t} + Ke^{\lambda t} + P\lambda e^{\lambda t}$

Here we go: $X' = AX$.
So $Kt\lambda e^{\lambda t} + Ke^{\lambda t} + P\lambda e^{\lambda t} = A(Kte^{\lambda t} + Pe^{\lambda t})$
$\Rightarrow$ $(AK - \lambda K)te^{\lambda t} + (AP - \lambda P - K)e^{\lambda t} = 0$ (rearranged to lin indep pieces)

Since the pieces are lin indep, the coefficients on the left must equal the coefficients on the right. Thus we need:
$(AK - \lambda K) = 0$ and $(AP - \lambda P - K) = 0$

These rearrange to $(A-\lambda I)K = 0$ and $(A-\lambda I)P = K$,
and that's what we must solve.

We must solve $(A-\lambda I)K = 0$ and $(A-\lambda I)P = K$.

The left-hand piece is the exact equation we use when we are looking for an eigenvector, so K is in fact an eigenvector for our eigenvalue λ.

The second equation, then, is just a system of equations where we know K and need to find P.
So K is just an eigenvector of A,
and P is a vector that yields K

<u>Back to our example:</u>

K is an eigenvector of $\lambda=-3$, which we knew to be $K = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Now we must solve the system $(A-\lambda I)P = K$.

$A-\lambda I = \begin{bmatrix} -4-\lambda & -1 \\ 1 & -2-\lambda \end{bmatrix} = \begin{bmatrix} -4+3 & -1 \\ 1 & -2+3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, so we must solve

$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which expands to $-p_1 - p_2 = 1$ and $p_1 + p_2 = -1$.

Those two equations are equivalent, so we just need to solve one of them, and any answer (other than 0 & 0) will work. Let's simply choose -1 & 0, so

$P = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Thus $X_2 = Kte^{\lambda t} + Pe^{\lambda t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{-3t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}e^{-3t}$

Finally, $X = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}. + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{-3t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}e^{-3t}\right)$

Here's one for you to try. My answer is on the next page.

Example 2. Solve $X' = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix}X$

Answer for previous page

Example 2: $X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}. \quad + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} e^{2t} \right)$

Distinct Complex Eigenvalues

Right off, let me say that we will never see repeated complex eigenvalues, since that would require at least a 4x4 system, and we are sticking with 2x2 systems.

Some facts about this situation

1. As we've seen before: If the DE has real coefficients, then complex eigenvalues appear in conjugate pairs. That is, both a+bi and a-bi. We usually denote these as λ and $\bar{\lambda}$. (It doesn't matter which you call which, but I usually take the a+bi as λ and the a-bi as $\bar{\lambda}$.

2. If λ is a complex eigenvalue, and K is its eigenvector (it will generally be complex as well), then $\bar{\lambda}$ (the conjugate) will have $\bar{K}$ (the conjugate of K) as its eigenvector.

3. So a general solution to this LHS could be: $X = c_1 K\exp(\lambda t) + c_2 \bar{K}\exp(\bar{\lambda} t)$. But we will not usually write it that way.

4. We prefer to rewrite the general solution as follows:
   Let's break K down into K = R + Si, the real and imaginary part of K. Both R & S are real.
   Then

$$X = c_3[R\cos(bt) - S\sin(bt)]e^{at} + c_4[R\sin(bt) + S\cos(bt)]e^{at}.$$

Example 3. Solve the system $x' = 2x - 5y$
$y' = 2x - 4y$

I'm assuming you can get the eigenvalues now. We get $\lambda = -1 \pm i$.

We need the eigenvector for $\lambda = -1+i$.

$$A-\lambda I = \begin{bmatrix} 2-\lambda & -5 \\ 2 & -4-\lambda \end{bmatrix} = \begin{bmatrix} 2-(-1+i) & -5 \\ 2 & -4-(-1+i) \end{bmatrix} = \begin{bmatrix} 3-i & -5 \\ 2 & -3-i \end{bmatrix}$$, so we must solve

$$\begin{bmatrix} 3-i & -5 \\ 2 & -3-i \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$. The two equations this gives are

$(3-i)c_1 + -5c_2 = 0$ and $2c_1 + (-3-i)c_2 = 0$.

It is not easy to tell, but these two equations are equivalent, so we only need to solve one of them, and we only need one solution to it. So let's solve the first.

$(3-i)c_1 + -5c_2 = 0$.

Remember our shortcut for solving? We swap the coefficients and negate one.

i.e. $c_1 = 5$ and $c_2 = 3-i$. That means our eigenvector is $\begin{bmatrix} 5 \\ 3-i \end{bmatrix}$.

We break that down into R+Si as follows: $\begin{bmatrix} 5 \\ 3-i \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}i$.

i.e. $R = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ & $S = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Also, since $\lambda = -1+i$, we have $a = -1$ and $b = 1$.
Now we plug into the solution.

$$X = c_3[R\cos(bt) - S\sin(bt)]e^{at} + c_4[R\sin(bt) + S\cos(bt)]e^{at}$$

$$= c_3[\begin{bmatrix} 5 \\ 3 \end{bmatrix}\cos(1t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(1t)]e^{-1t}$$

$$+ c_4[\begin{bmatrix} 5 \\ 3 \end{bmatrix}\sin(1t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(bt)]e^{-1t}.$$

Note: ALWAYS use the notes above while doing these. Don't waste your time memorizing the pieces.

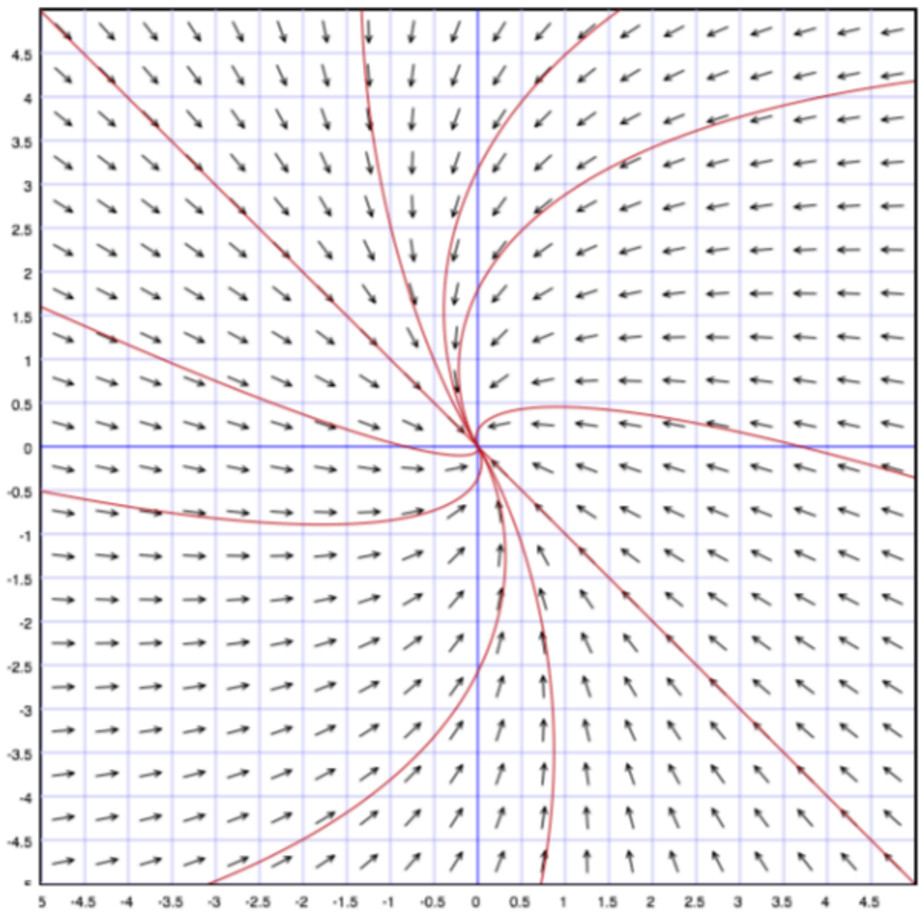
Lastly, what about the phase planes for these examples?

Repeated Real Eigenvalues

We would still have the straight-line solution for the first eigenvalue, just like we did last time. The second solution does NOT correspond to a straight-line solution. Rather it just contributes to the curving of the solution curves.

Notice that each piece has $e^{\lambda t}$ in it, so each piece will go toward either zero of $+\infty$, depending on whether λ is negative or positive.

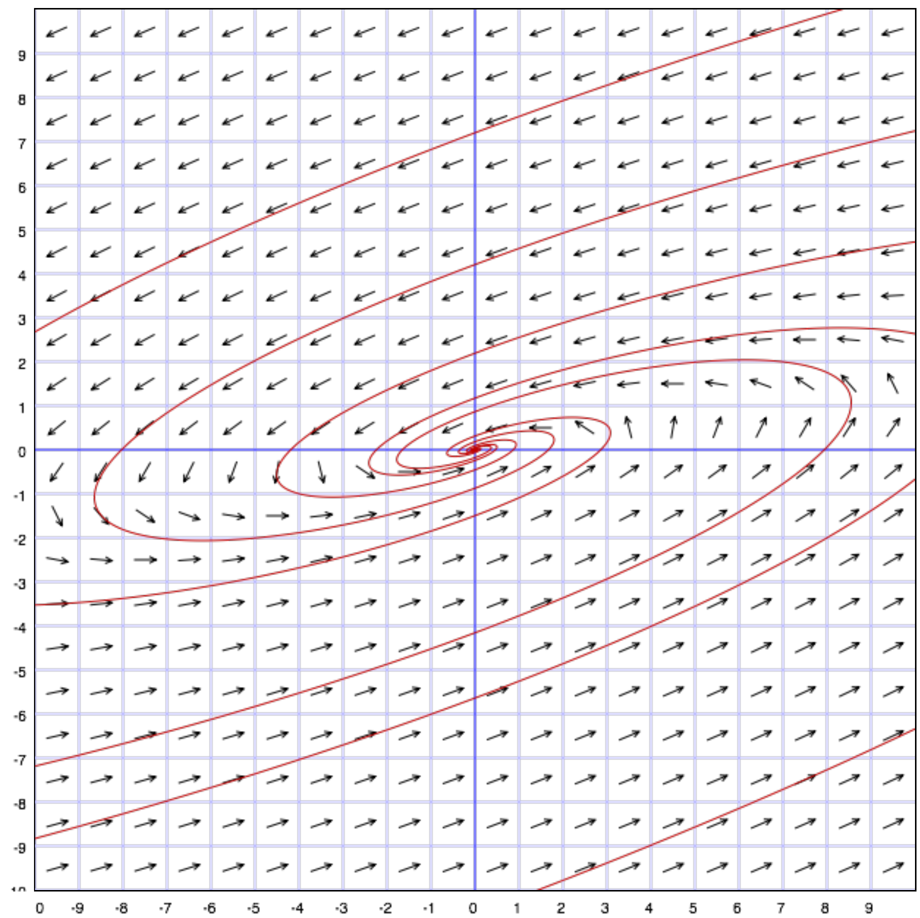
The solution curves would look like this:



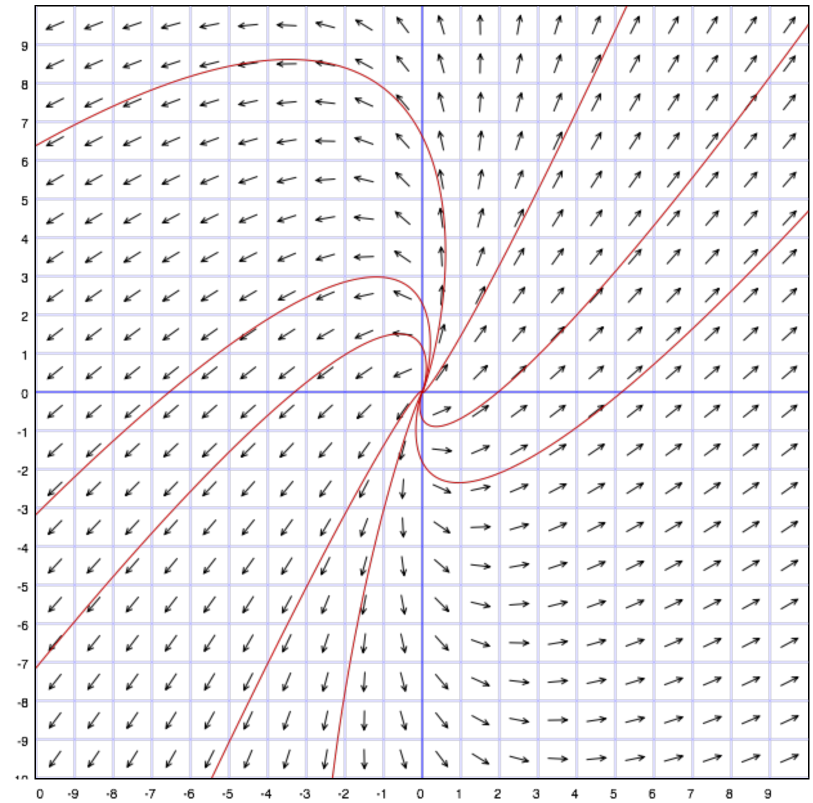
This happens to be the phase plane for our Example 1 above in this document.

Now for the phase planes for complex eigenvalues. We know that the sines and cosines are periodic, so these curves are going to rotate around the origin. But the in the solution will cause the curves to move inward or outward, depending on with the real part of the eigenvalue (what we call a of the a±bi) is negative or positive. In fact, the a part can be zero, as well, in which case the solutions will rotate around right back on top of themselves.

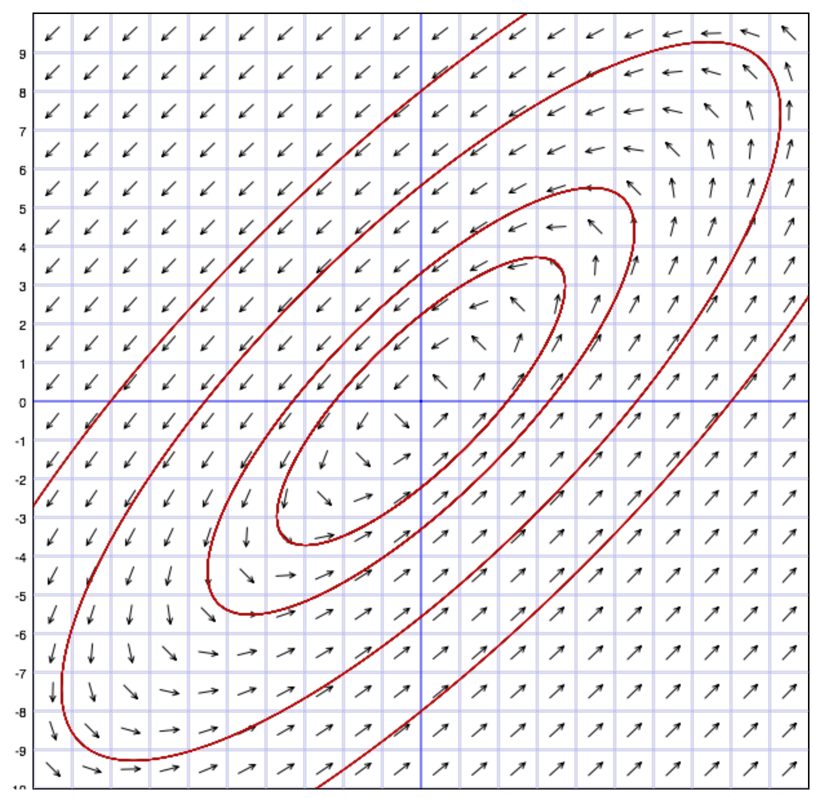
Here's an example of the spiraling inward variety, so the a-value is negative.



Here’s an example of the spiraling outward variety, so the a-value is positive.



Here’s an example of the periodic cycling variety, so the a-value is zero.



Homework for this section

8.2 Do problems 19 & 20.

Email me your solution to #20.
For convenience, instead of c_1 & c_2, you can just type c1 and c2,

and instead of things like $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$, you can just type <5,3>,

and for e^{at}, you can type e^at.

Also Do problems 33 & 34.

Email me your solution to #34.
Same conveniences as above.

If you find it easier to scan/photo your work and send that to me, you are welcome to do that. My email requests above are intended to make your life a little easier, not more difficult!